# Assessing the extent of strategic manipulation for the average voting rule *

Régis Renault[†]and Alain Trannoy[‡]

December 22, 2004

**Abstract**

The average voting procedure reflects the weighted average of *expressed* opinions in [0,1]. Participants typically behave strategically. We evaluate the discrepancy between the average taste and the average vote. If the population is sufficiently large, it is possible to contruct approximations of both the average vote and the average taste which may be readily compared. We construct upper and lower bounds for the limit average vote that depend on the limit average taste. If the average taste is central enough, the range of possible values for the average voting outcome is narrower than the corresponding range for majority voting. For instance, if the average taste is at 1/2, the limit equilibrium outcome is this value plus or minus roughly .2, whereas the median maybe anywhere in the [0,1] interval. Results are robust to the introduction of incomplete information.

**Keywords:** Average voting, Nash equilibrium, Strategic Bias.
(**JEL**: D74, H41, I22)

*We thank Hervé Moulin for a useful suggestion. We should also like to thank Francis Bloch, Alessandra Casella, Nicolas Gravel, Michel Le Breton and William Thomson for their comments on earlier versions of this paper. The usual caveat applies.

[†]Université de Cergy-Pontoise, THEMA, 33 Bd du Port, 95011 Cergy-cedex, France, Institut Universitaire de France, (e-mail : regis.renault@eco.u-cergy.fr ).

[‡]EHESS, GREQAM-IDEP, Vieille Charité, 2 rue de la Charité, 13002 Marseille, France, (e-mail : trannoy@ehess.univ-mrs.fr).

# 1 Introduction

In response to the Gibbard-Satterthwaite theorem (Gibbard (1974), Satterthwaite (1975)) much attention has been devoted to truthful implementation. Yet, strategyproofness or less restrictive criteria, like Nash or Bayesian incentive compatibility, rule out some simple voting rules that are commonly used in practice as, for instance, plurality in political elections or Borda count in committee decisions. When dealing with such rules that allow for strategic manipulation, it is useful to assess the extent of the manipulation by performing a comparison of the outcome of strategic behavior with that of sincere voting. The latter is a natural benchmark since one might expect those who chose these rules at a constitutional stage to view them as appropriate modes of preference aggregation. Here we present a method for evaluating the extent of the distortion introduced by strategic manipulation, in other words the strategic bias, in the case of the average voting rule.

It is a very simple voting scheme that implements a weighted arithmetic mean of votes. Several countries have quite recently adopted procedures for allocating public funds, that may be described by a weighted average vote. In Spain, tax payers may earmark up to 0.5% of their income tax to the catholic church or to non-governmental organisations and similar provisions can be found in Italy or Portugal. In Canadian provinces of Ontario and Saskatchewan, there are publicly financed separate school boards for Catholic schools along with the public school boards; households may choose which school system receives their property taxes. In France, high schools, colleges and universities are partly financed by a "training tax" that firms must pay[1], although they may decide on its allocation among different teaching institutions or training programs. Typically, firms and more particularly small ones choose to finance only one institution. These tax mechanisms are formally equivalent to weighted average voting rules. If there are only two possible uses of public funds, the vote of a tax payer is the fraction of her taxes that she chooses to allocate to one of them. Then the outcome of the vote (the proportion of public funds going to either use) is a weighted average of the votes, where the weight of each voter is her share in total tax contributions. Although

[1]Payrolls are taxed at a 0.5% rate, which yields a revenue of 1.2 billion in 2002.

the weights represent the individual share in total wealth or in total tax contribution in all actual applications of the average vote that we are aware of, the interpretation of the weights may be broader[2]. For instance, if each voter represents a group (household, constituency, country...), the weight may be the share of this group in the overall population.

Although there are numerous examples of its application, the average voting rule has only attracted limited attention. We know that sincere average vote yields an efficient outcome, if agent's preferences are Euclidean. In this case, the set of Pareto outcomes is identical to the set of weighted average votes. If there at least five agents, agents have Lipschitz utility functions and the voting space is multidimensional, the average voting rule is shown to be the unique anonymous and unanimous voting rule that satisfies a weakening of strategy-proofness in large voting problems (Ehlers and al. 2004). Renault and Trannoy (2003b) present an axiomatization of the true weighted average vote in order to shed some light on its normative properties as a benchmarck. Bilodeau (1994) in his study of tax-earmarking institutions shows that leaving the spending decisions in the hands of individuals yields a unique non-cooperative equilibrium in the core. Renault and Trannoy (2003a) exhibit circumstances where the average rule may be more suited to protect minorities than majority voting, taking into account the strategic behavior of voters.

In the average voting game considered here, individuals choose an alternative in the [0,1] interval and preferences are supposed to be single-peaked in order to allow an easy comparison of the outcome of the game with that of majority voting. In particular we compare the strategic bias in average voting with the discrepancy between the average and the median taste, which serves as a benchmark. Indeed, a distortion from the average taste that would exceed that obtained by using the strategyproof majority rule would seem particularly unsatisfactory.

The game is studied both in a complete and incomplete information context. The agent's

[2] Average voting is here considered as a direct democracy mechanism. There are a number of papers in the political economy literature that describe the democratic political process as achieveing a compromise modeled as a convex combination of the political platforms of the various parties (see Alesina and Rosenthal, 1995, 1996). The weights depend on the distribution of votes among parties and there is therefore some formal equivalence with average voting, which is exploited in Gerber and Ortuno-Ortin (1998).

Nash equilibrium behavior is typically to vote either 0 or 1, which is in tune with the empirical evidence for the training tax. The characterization of the equilibrium outcome is quite clean and allows for an easy comparison with the outcome of a majority vote. It is less obvious how it relates to the average taste. However, if the population is sufficiently large, it is possible to construct approximations of both the average vote and the average taste that may be readily compared: the limit equilibrium allocation is characterized by a simple fixed point relation involving a function that, for all possible levels of the allocation $y$, indicates the expected relative weight of those who favor an outcome above $y$; the integral of this same function approximates the average taste for a large population. We further show that the fixed point formula also approximates the equilibrium outcome when tastes are private information to voters, while weights remain common knowledge.

These approximation results allow us to construct upper and lower bounds for the limit average vote that depend upon the limit average taste. Average voting prevents the outcome from being too extreme when the average opinion is central. It restricts the range of the social outcome more than the majority rule. For instance, if the average taste is at 1/2, the limit equilibrium outcome is within plus or minus roughly .2 around this value, whereas the median may lie anywhere in the [0,1] interval. This result shows that in a very polarized society where a large number of voters have extreme opinions 0 or 1, average voting is better suited than majority voting for achieving a compromise, i.e. an outcome different from the most extreme opinions[3]. Moreover, the strategic bias is at most .21, again reached for an average taste of one half. When the average taste is closer to one of the boundaries of the choice space, namely the average taste is smaller than 1/4 or larger than 3/4, the strategic bias may be more important than the gap between the average and the median taste. Indeed, the strategic power is much stronger for those who favor an outcome in the center of the choice space; in this case they are all located on one side of the average taste.

We gather what is known about the average voting game in a first introductory section.

[3]See Border and Jordan (1983), for a related formulation of the uncompromising nature of the median rule. The majority rule does not exploit the continuity of the choice space which would allow for such a compromise.

A limit approximation of the average taste is provided in Section 3, as well as the main results regarding the approximation of the Nash outcome in a complete information setting for large populations, the strategic bias and the comparison with majority voting. The case of incomplete information is the topic of section 4, where it is proved that the outcome of the game may be approximated by the formula exhibited in section 3, implying that all the results found in the context of complete information apply to the context of incomplete information as well. Section 5 concludes. Proofs of results are gathered in the appendix.

# 2 Preliminaries

We start with a brief description of the average voting game along with an overview of existing results. The model here is adapted from Renault and Trannoy (2003a). There are $n$ voters with singlepeaked preferences over the choice space which is the unit interval. Each voter $i$ chooses a vote denoted $s_i$ in $[0, 1]$ and voting involves no costs. Agents cast their votes simultaneously. The allocation is then defined by

$$y = \sum_{i=1}^{n} w_i s_i, \tag{1}$$

where $w_i \geq 0$ is the relative weight of voter $i$, for any $i$, and $\sum_{i=1}^{n} w_i = 1$.

To understand how agents behave in a Nash equilibrium, it is useful to describe best responses. Other player's choices are only relevant to player $i$ through an aggregate vote. Let $S_{-i}$ be the weighted sum of votes by voters other than $i$, that is, $S_{-i} = \sum_{j \neq i} w_j s_j$. Then agent $i$'s best response is defined by

$$s_i(b_i, S_{-i}) = \begin{cases} 1 & \text{if } b_i - S_{-i} > w_i \\ \frac{b_i - S_{-i}}{w_i} & \text{if } 0 \leq b_i - S_{-i} \leq w_i \\ 0 & \text{if } b_i - S_{-i} < 0 \end{cases} \tag{2}$$

If the aggregate vote by others is below the bliss point $b_i$ two situations are possible, depending on the size of the discrepancy. If it is larger than agent $i$' s weight, it is optimal to pick the largest possible vote which is 1. If the difference is smaller, agent $i$'s weight in the average vote, $w_i$ enables him to make up for the discrepancy, in which case he obtains his exact bliss point as the final outcome. If the aggregate vote by others yields a value that is above the bliss point, it is optimal to vote 0 since any non zero vote would make the situation worse.

The characterization of the equilibrium outcome requires that individuals be ranked according to decreasing values of $b_i$. Let us define:

$$W_i = \sum_{j=1}^{i} w_j.$$

Now let

$$i^* = \min\{i \in \{1, ...., n\}; W_i \geq b_{i+1}\}$$

with $b_{n+1} = 0$.

In order to state existing results regarding the equilibrium of the average voting game, we first give two definitions. We define the median of a finite set of real numbers $A$ with $N$ elements as the smallest number $med(A) \in A$ that satisfies

$$\frac{1}{N}\#\{a \in A : a \leq med(A)\} \geq \frac{1}{2} \text{ and } \frac{1}{N}\#\{a \in A : a \geq med(A)\} \geq \frac{1}{2} \tag{3}$$

If $N$ is odd, condition (3) defines a unique number while if it is even, there are 2 such numbers. We adopt the convention that the median is the smallest.

We also define a strong Nash equilibrium following Auman (1959). Let $C$ be a coalition i.e. a subset of the set of voters, and let $\bar{C}$ the complement of $C$ in the set of voters. The $n$-tuple $s^* = (s_1^*, ..., s_n^*)$ is a pure-strategy strong Nash equilibrium if for all coalitions $C$ and for all $s \in [0, 1]^n$,

$$u_i(\sum_j w_j s_j^*) \geq u_i(\sum_{j \in cC} w_j s_j + \sum_{j \in \bar{C}} w_j s_j^*) \text{ for all i} \in C$$

where $u_i$ denotes voter $i$'s utility.

**Proposition 2.1** *The average voting game has a Nash equilibrium. The Nash equilibrium allocation is unique and may be described by the following two equivalent formulas:*

$$y^* = \min\{b_{i^*}, W_{i^*}\}. \tag{i}$$

$$y^* = med(b_1, ..., b_n, W_1, ..., W_{n-1}) \quad \text{(ii)}$$

*Furthermore any Nash equilibrium is strong Nash.*

The two characterizations are established in Renault and Trannoy (2003a) and the argument showing that any Nash equilibrium of this game is strong Nash can be found in Bilodeau (1994).[4] Since, as Bernheim, Peleg and Whinston (1987) point out, any Strong Nash equilibrium is coalition-proof, any Nash equilibrium of this game is also coalition-proof.

Characterization (ii) bears a striking resemblance with the extended median of Moulin (1980)[5] who shows that a social choice rule, $f$, is peak-only (i.e. the sole relevant information about preferences for the social choice rule is the list of bliss points), strategyproof, anonymous and efficient if and only if there exist $n-1$ parameters in $[0,1]$, $a_1, ..., a_{n-1}$ such that for all profiles of single peaked preferences, $R$,

$$f(R) = med\{b(R_1), ...., b(R_n), a_1, ..., a_{n-1}\}.$$

In this definition, the parameters $a_1, ..., a_{n-1}$ are independent of the preference profile under consideration while the parameters $W_1, ..., W_{n-1}$ typically depend on how the bliss points are ranked. Note that, however, if weights are identical, characterization (ii) is an extended median with parameters $\{\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}\}$ no matter what the ordering of bliss points turns out to be. To our knowledge this voting procedure provides the first illustration of how the parameters of an extended median may have an economic interpretation, namely as population shares. Here the uniform distribution corresponds to an equal treatment of individuals. The careful reader will notice that the parameters $\{\frac{1}{n}\ , \frac{2}{n}, ....\frac{n-1}{n}\}$ are related to the outcome of the social choice for some specific profiles of preferences. More precisely it is easily checked that $\frac{n-i}{n}$ is the outcome of the extended median rule when $n-i$ (respectively $i$) individuals have 1 (resp . 0) as bliss points. It is also the average taste for this profile. Since, in this case, no individual wants to manipulate his vote when faced with the average

[4]The uniqueness result is reminiscent of Gerber Ortuno-Ortin (1998) who find, in a similar game with a continuum of agents that there exists a unique strong Nash equilibrium in which voters use cut points strategies. The unique strong Nash allocation is also a core allocation, when the core definition keeps in line with the literature on voting games with the majority rule (see for instance Ordershook 1986).

[5]See also Sprumont (1995) for a detailed survey of this material.

voting procedure, the outcome of this procedure will coincide with the true mean of bliss points. The same kind of remarks applies for the nonanonymous case at the price of some additional complexity.

We now turn to our main focus, which is an evaluation of how much strategic behavior distorts the outcome of the vote from the average taste.[6] To this end, the remainder of the paper considers the average voting game with a large population of voters. For the following, the reader needs to remember that in equilibrium, all those with bliss points strictly above the equilibrium allocation vote 1 while all those with bliss points strictly below the equilibrium allocation vote 0.

# 3 Assessing the strategic bias with a large population of voters.

## 3.1 Inferring the average taste from aggregate data

Although we assume in this section that weights and bliss points are common knowledge for the voters, this common knowledge may not be shared by an outside observer. He may only have some aggregate knowledge, namely, he does not know any more than the joint probability distribution of bliss points and weights. From this point of view, it is relevant to provide an approximation of the equilibrium outcome which requires only the knowledge of this probability distribution when the population is large enough. Correlation between bliss points and weights is allowed since it is likely present in all practical applications.

To proceed with the limit argument, we need to derive a simple expression for the limit of the weighted average taste. Bold characters denote random variables. A vote with $n$ participants is given by $n$ independent draws from a probability distribution $\mathcal{P}$ defined on $[0,1]\ \times\mathcal{R}_{++}$ admitting a continuous density. For each player $i$, the first component is his bliss point $\mathbf{b}_i$ and the second component is his absolute weight $\boldsymbol{\omega}_i$, which contrary to relative weight $w_i = \omega_i / \sum_{i=1}^{n} \omega_i$ is not restricted to be in [0,1].[7] Let $F$ denote the marginal c.d.f. of

[6] It comes as no surpprise that the average voting rule is not Nash implementable, since it does not satisfy the monotonicity criterion of Maskin (1977 and 1985).

[7] Specifying absolute weights is convenient because, contrary to relative weights, they may be drawn

$\mathbf{b}_i$ and let $J(. \mid b_i)$ be the conditional c.d.f. of $\boldsymbol{\omega}_i$. We further assume that the conditional distributions have finite mean for all values of conditioning variables. Let $\mu(b)$ denote the conditional mean and $\overline{\mu}$ denote the unconditional mean of $\boldsymbol{\omega}_i$. We now define the decreasing function $H$ on $[0,1]$ as follows

$$H(y) = \overline{\mu}^{-1} \int_y^1 \int_0^{+\infty} \omega dJ(\omega \mid b_i) dF(b_i) = \overline{\mu}^{-1} \int_y^1 \mu(b_i) dF(b_i). \tag{4}$$

This function is decreasing from $H(0) = 1$ to $H(1) = 0$. It measures the expected relative cumulative weight of individuals with bliss points in excess of $y$. In the special case where weights are independent from bliss points, we have $H(y) = 1 - F(y)$.

In the general case, the function $H$ may be also related to $F$ thanks to a concentration curve. Whenever we plot shares of a variable $X$ against quantiles in the distribution of a variable $Y$, the result is called a concentration curve for $X$ with respect to $Y$ (see Lambert (1993) p.37). Now define $G$ as the function that, for all $y \in [0,1]$, maps $1 - F(y)$ into $H(y)$ so that

$$H(y) = G(1 - F(y)).$$

Note that $1 - F(y)$ is the expected cumulative proportion of the population with bliss points above $y$, while $H(y)$ is the expected cumulative relative weight of this subpopulation. The function $G$ may therefore be viewed as the concentration curve for weights with respect to bliss points.

The function $H$, which plays a critical role in the paper, may first be used to derive a simple characterization of the limit of the average taste as the population becomes large.

**Proposition 3.1** *The weighted average taste for n voters* $\nu_n = \sum_{i=1}^n w_i b_i$ *converges to* $\nu = \int_0^1 H(b)db$ *with probability 1 when n goes to infinity.*

This result generalizes the well known formula for the expected value of a random variable with c.d.f. $F$ as $\int_0^1 1 - F(t)dt$. The generalization we consider here allows for different realizations of a random variable to be weighted differently.

independently, and we may therfore resort to law of large numbers arguments in the proofs.

As we now show, the function $H$ may also be used to construct an approximation for the Nash outcome of average voting.

## 3.2 Inferring the Nash outcome from aggregate data

Let $F_n$, $J_n(. \mid b_i)$ $\mu_n(b_i), \overline{\mu_n}$ and $H_n(y)$ be the empirical counterparts of $F$, $J(. \mid b_i)$, $\mu(b_i), \overline{\mu}$ and $H(y)$ for the $n$-players game. It is readily shown that

$$H_n(y) = \overline{\mu}_n^{-1} \int_y^1 \mu_n(b_i) dF_n(b_i) = \frac{\sum_{i=1}^n \omega_i I(b_i \geq y)}{\sum_{i=1}^n \omega_i},$$

so that it measures the relative cumulative weight of agents with bliss points of at least $y$ (where $I$ denotes an indicator function). This empirical cumulative weights function $H_n$ is a decreasing step function which is left continuous. Points of discontinuity correspond to realized bliss points $\{b_i\}_{i=1}^n$ and the jump at $b_i$ measures the relative weight of individuals with bliss point $b_i$. From now on, $\mathbf{y}_n^*$ denotes the equilibrium allocation when the population size is $n$ and $y^*$ denotes the unique solution to

$$y^* = H(y^*) = G[1 - F(y^*)]. \tag{5}$$

We have:

**Proposition 3.2** *The sequence* $\{\mathbf{y}_n^*\}$*converges to* $y^*$ *with probability* 1.

In the limit, votes are concentrated at the extremes, 0 or 1, so that the outcome is given by the cumulative weight of those voting 1, $H(y^*)$, and it is also equal to the bliss point of the pivotal individual, $y^*$.

An interesting special case of the above result is when weights $\boldsymbol{\omega}_i$ are independent of bliss points $b_i$. Then, $\mu(b_i) = \overline{\mu}$ for all $b_i$. Here the game is anonymous in expectation in the sense that the expected weight of individuals is the same no matter what their tastes might be. Therefore, from (4), $H(y) = 1 - F(y)$. Thus $y^*$ is defined by the simple fixed point relation

$$y^* = 1 - F(y^*).$$

The approximation that we have derived for the average taste and the Nash outcome are quite simple and may be used to evaluate the extent of the strategic bias.

### 3.3 Strategic bias

We now investigate by how much the target may be missed. To this end we establish the following proposition pertaining to the sign and magnitude of the strategic bias, i-e, the gap betweent the average voting outcome and the average vote $\nu = \int_0^1 H(b)db$.

**Proposition 3.3** *(i) If $H$ is strictly convex, then $y^* \geq \nu$, while if $H$ is strictly concave the inequality is reversed.*
*(ii)*$1 - \sqrt{1-\nu} \leq y^* \leq \sqrt{\nu}$.

Result (i) provides some hint as to how the sign of the strategic bias is related to the skewness of the weighted distribution of bliss points: concavity or convexity of $H$ corresponding to the extreme cases of a monotonically increasing or decreasing density. If the distribution is skewed downwards so that $H$ tends to be convex, the bias should be expected to be upwards. This is because the mean is close to zero so that those who favor an outcome below the mean have only a limited ability to distort their vote; the most they can do is to vote zero. By contrast, those who favor an outcome above the mean have bliss points remote from 1 so that they may hugely exagerate their taste.

The second result provides bounds on the value of the Nash outcome as a function of the average taste. As shown by the proof of the above proposition, the obtained bounds are tight in the sense that it is always possible to specify a continuous function $H$ such that the average voting outcome is arbitrarily close to one of these bounds. The largest possible interval is obtained for a mean of 1/2, in which case the lowest possible Nash outcome is $1 - \sqrt{1/2}$ and the largest is $\sqrt{1/2}$. Thus, average voting guarantees that if the average taste is moderate, the collective choice cannot be too extreme. In this particular configuration, the allowed interval for the average vote outcome is symmetric with respect to the average taste, a property that is lost when the average taste is closer to 0 or 1. Indeed, players who have an opinion farthest from the boundaries of the choice space can pull the outcome towards their preferred opinion very effectively, since they may hugely distort their expressed opinion from the true one by casting an extreme vote. Thus, for instance, if the average taste is below 1/2, the largest potential upward bias is greater than the largest potential downward bias.

Since the set of all allowed values for the average voting outcome is fairly large, one may wonder whether it is narrower than the set of allowed values for the outcome of the majority vote. Here, the relevant majority vote is one where each voter is weighted as in the weighted average taste, so that the outcome is a weighted median $m_n$ defined as follows

$$\sum_{i/b_i \leq m_n} w_i \geq \frac{1}{2} \text{ and } \sum_{i/b_i \geq m_n} w_i \geq \frac{1}{2}, \tag{6}$$

with the convention that if there are two such numbers the smallest will be selected.

It is straightforward to establish that when $n$ converges to infinity, $m_n$ converges with probability 1 to $m$ defined by

$$H(m) = 1/2.$$

The proof of the following proposition is similar to the proof of (ii) in the previous proposition.

**Proposition 3.4** $Max(0, 2\nu - 1) \leq m \leq Min(1, 2\nu)$.

The comparison of the intervals in Propositions 3.3 *(ii)* and 3.4 leads to mixed conclusions about the merits of average voting relatively to the weighted majority vote in reflecting the average taste. It is readily verified (see Figure 1 for an illustration) that the allowed interval for the average outcome is strictly included in that for the weighted median as long as the average taste is strictly between 1/4 and 3/4. The advantage of the average voting rule over majority voting is especially telling in this case, since the magnitude of the interval and therefore of the potential discrepancy between the average taste and the outcome of the vote is maximal. For a smaller average taste, both the lower and upper bound on $y^*$ are strictly above the bounds on the median. The opposite configuration prevails for an average taste above 3/4. In these two cases, neither interval is a subset of the other[8]. Then, as the average taste approaches either end point, no clear-cut conclusion about the relative merit of the average vote may be drawn. Nevertheless, the potential distortion from the average taste is

[8]It can be mentioned that the length of the interval for the average vote is smaller than that of the weighted majority vote for $\nu \in [0.157, 0.843]$.

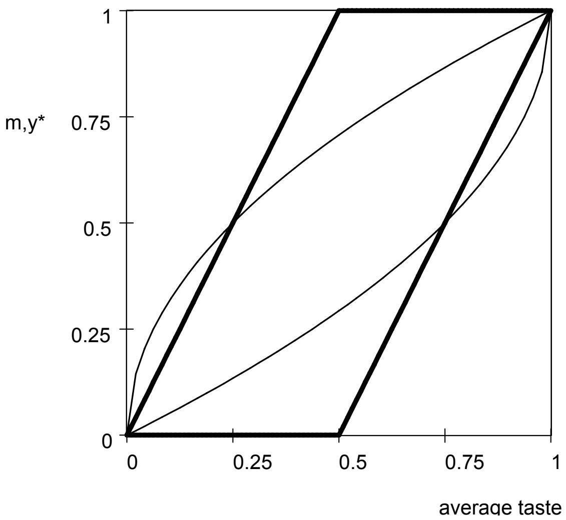


Figure 1: The thick lines (resp. thin lines) represents the allowed interval for the majority (resp. average) outcome in function the average taste.

less of an issue as the average gets closer to one end, since the allowed intervals for both the average vote and the majority vote outcomes become smaller.[9]

We conclude that if the average taste is located close to 0 or 1, the average voting rule is hampered by the excessive strategic power of voters whose bliss points are in the center. One way to remedy this is to limit the set of possible votes as is done in actual applications of the average voting rule in Italy, Portugal and Spain.

Finally it should be noted that a comparison of the allowed interval for the two possible outcomes is relevant only if the policy-maker has a poor information about the underlying distribution of bliss points and weights that generates the function $H$. It may well be that, for a specific distribution, the outcome of the average vote does worse than the outcome of majority voting in approximating the average taste even though the latter falls in $[1/4, 3/4]$. Consider for instance the case of a population comprised of two subgroups: 60% are drawn according to a continuous uniform probability distribution on the support $[0, .4]$ and there is a mass point of 40% at .4. The outcome with the average vote is .4, whereas the median is about .33 which is closer to the mean .28.

[9]This, however, is not true in relative terms since the ratio of the largest possible average vote outcome to the mean tends to infinity as the mean tends to zero.

# 4 Extension to an incomplete information setting

All results of the previous section have been obtained assuming that weights and bliss points are common knowledge. When the number of individuals involved in the average voting game increases, it seems more appropriate to consider an incomplete information setting. An incomplete information version of the average voting game is now considered. It turns out, as shown in subsection 4.3, that the limit outcome does not depend upon which information structure is selected.

## 4.1 The Bayesian Game

It is now assumed that two agents with identical bliss points have identical preferences over the set of allocations. Let $V$ be a continuous function defined on $[0,1]^2$ which is such that $V(y, b_i)$ is agent $i$'s utility associated with the allocation $y$ if his bliss point is $b_i$. The realization of $b_i$ is private information to agent $i$ while the distribution functions as well as other parameters of the game and in particular the weights are common knowledge. Thus, from the point of view of other agents, agent $i$'s bliss point is distributed according to a probability distribution conditional upon the weight realization, with a c.d.f. denoted $F(. \mid \omega_i)$. A pure strategy profile, $s$, is a mapping from $[0,1]^n$ into $[0,1]^n$ where $s_i(b_i)$ is agent $i$'s vote when agent $i$'s bliss point is $b_i$. We look for Bayesian equilibria.

**Definition 4.1** *A pure strategy Bayesian equilibrium is a strategy profile $s^*$ which satisfies for all $i \in N$*

$$s_i^*(b_i) \in \arg\max_{s \in [0,1]} E_{\mathbf{S}^*_{-i}}(V(\mathbf{S}^*_{-i} + w_i s, b_i)) \tag{7}$$

*with $\mathbf{b}_{-i} = (\mathbf{b}_1, ...., \mathbf{b}_{i-1}, \mathbf{b}_{i+1}, ...., \mathbf{b}_n)$ and $\mathbf{S}^*_{-i} = \sum_{j \neq i} w_j s_j^*(\mathbf{b}_j)$.*

The following proposition deals with the issue of existence.

**Proposition 4.1** *There exists a Bayesian equilibrium in pure strategy.*

We now turn to a characterization of the equilibrium. To this end, we make the following additional assumptions about the utility function.

**Assumption 4.1 .** *The function $V$ is continuously differentiable. Let $V'$ be the partial derivative of $V$ with respect to $y$; it is assumed to be strictly decreasing in $y$ and strictly increasing in $b$ on* $[0,1]$.

Thus $V$ is strictly concave in the allocation and the marginal utility of $y$ is all the larger that the bliss allocation is large. We now establish

**Proposition 4.2** *(i) Each component $s_i^*$ of any pure strategy Bayesian equilibrium $s^*$ is continuous and increasing on* $[0,1]$. *(ii)There exits two pivotal bliss points $\underline{b}_i$ and $\overline{b}_i$ such that $s_i^*(b_i) = 0$ if and only if $b_i \in [0, \underline{b}_i]$ and $s_i^*(b_i) = 1$ if and only if $b_i \in [\overline{b}_i, 1]$.*

Typically agents with low bliss points vote 0 and agents with high bliss points vote 1. This type of behavior characterized by an overstatement of one's taste is very similar to that exhibited in the complete information setting.

The next subsection presents a characterization of the limit of the equilibrium outcome when the number of voters goes to infinity.

## 4.2 Approximating the Bayesian outcome

We now consider a sequence of incomplete information voting games and we show that for a large enough population, the outcome of the game may be approximated by a simple fixed point relation depending upon $n$. In the following analysis, the sequence of realizations of weights $(\omega_1, ..., \omega_n)$ is assumed to satisfy

**Assumption 4.2** *Let $\overline{w}_n = \max_i w_i$ where $w_i = \omega_i / \sum_{j=1}^n \omega_j$. The sequence $\{\overline{w}_n\}$ converges to* 0.

From Proposition 4.2 there is an interval of types whose vote is strictly between 0 and 1 which is denoted $[\underline{b}_{in}, \overline{b}_{in}]$ in an $n$-players game. For any $n$, let us define $\underline{b}_n \equiv \min_i \underline{b}_{in}$ and $\overline{b}_n \equiv \max_i \overline{b}_{in}$. The following lemma shows that the interval between these two bounds shrinks as the population size becomes large.

**Lemma 4.1** *Under the above assumptions,* $\lim_{n\to\infty}[\overline{b}_n - \underline{b}_n] = 0$.

The intuition behind this result is the following. Recall that from the point of view of each player, an equilibrium outcome for the $n$-players game denoted $\mathbf{y}_n^{**}$ is a random variable. The only thing he is certain of, is his own vote, $s_i$. An increase in $s_i$ causes the distribution of $\mathbf{y}_n^{**}$ to shift to the right. If his weight in the vote is large, he may wish to fine tune the distribution of the outcome by picking a vote strictly between 0 and 1. As his weight becomes smaller, it becomes more unlikely that such a fine tuning is desirable for him. Either he is better off leaving the distribution as it is by voting 0 or he chooses to throw all his weight into moving the distribution to the right.

Lemma 4.1 says that, in the limit, almost all agents choose an extreme vote. Then the average vote is approximately equal to the proportion of voters who vote 1 (i.e. those whose bliss point is on the right of $\overline{b}_n$). This result is now used to provide an approximation for the equilibrium outcome for $n$ large enough.

**Proposition 4.3** *Under the above assumptions, letting* $\widetilde{y}_n$ *be uniquely defined by*

$$\widetilde{y}_n = \widetilde{H}_n(\widetilde{y}_n) \tag{8}$$

*where* $\widetilde{H}_n(y) = \sum_{i=1}^n w_i(1 - F(y \mid \omega_i))$ *for all* $y \in [0, 1]$, *we have*

$$P \lim_{n\to\infty} (\mathbf{y}_n^{**} - \widetilde{y}_n) = 0$$

*for any sequence* $\{\mathbf{y}_n^{**}\}$ *of equilibrium outcomes.*

The value $\widetilde{H}_n(y)$ may be interpreted as the expected weight of those individuals who hold a bliss point above $y$ given the weights' realizations.

As in the limit result under complete information, the situation where the bliss point distribution is independent of weights yields a simple expression for the fixed point. Indeed, the function $\widetilde{H}_n$ melts down to $1 - F$ so that $\widetilde{y}_n = y^*$ for all $n$[10].

It is now shown that the two information structures actually yield very similar outcomes when the population is large.

[10]Note that convergence in that case would be with probability 1: in the beginning of the proof of Proposition 4.3 we could appeal to a strong law of large numbers argument because $\mathbf{C}_{1n}$ and $\mathbf{C}_{2n}$ are sums of i.i.d. random variables $I(b_i > y)$ which is not true in the general case where variables are only independent.

### 4.3 Irrelevance of the information structure for large populations

In the above analysis, we assume a particular realization of weights $(\omega_1, ..., \omega_n)$. Even though these weights are common knowledge for the voters, the observer may only have some aggregate knowledge of the weights vector, i-e., he does not know any more than the weight distribution. We now address the question of how the outcome of the game may be predicted by an observer who is unaware of the weight realizations and only knows the probability distribution $\mathcal{P}$.

The following lemma shows that Assumption 4.2 essentially always holds if the population is large.

**Lemma 4.2** *The sequence* $\{\overline{\mathbf{w}}_n\}$ *converges to 0 with probability 1.*

Therefore, the above results, Lemma 4.1 and Proposition 4.3 hold with probability 1. As under complete information it is possible to provide a limit characterization of the equilibrium outcome independent of $n$.

**Proposition 4.4** *The sequence* $\{\mathbf{y}_n^{**}\}$*converges to* $y^*$*with probability* 1.

The above proposition shows that in a large economy, the equilibrium outcome may be approximated by the same fixed point relation whether or not players are imperfectly informed about each other's taste[11]. It follows that our analysis of the strategic bias is independent of the information structure of the game.

## 5 Conclusion

The average voting rule is an instance of a voting procedure that is used in various contexts even though it is not immune to strategic manipulation. Typically, voters choose to cast extreme votes. Our focus here is on the extent of this strategic manipulation as it is reflected in the discrepancy between the outcome of the vote and the average taste. The outcome of the average voting game may easily be compared with the mean of the

[11] Although we drive this result assuming that voters know each other's weights, we conjecture that it would not be affected if weights are private information as well.

populations's true opinions when the population is large enough. It is possible to establish that the outcome of average voting lies in some interval containing the average taste. The strategic bias may then be evaluated by comparing this interval with the range of potential outcomes for majority voting. If the average taste is not too extreme, then the range of potential outcomes for average voting is included in the corresponding range for majority voting. For more extreme average tastes, neither voting procedure dominates the other but they both yield outcomes that may not be too remote from the average taste.

The approximation of the equilibrium outcome that is derived here only requires aggregate information, that is the joint distribution of weights and bliss points. Remarkably, under mild assumptions, the approximation formula is independent of the information structure of the voting game. The driving force for this result seems to be that the game is anonymous, in the sense that the only relevant information for each player is the distribution of other players' decisions. It raises the question whether the irrelevance of the information structure emerges for a larger class of games sharing this anonymity property.

The more general message that we wish to convey is that, although truthful implementation may be a desirable property for a social choice rule, imposing such a requirement may lead to an excessive impoverishment of social choice theory. A more sensible approach exemplified by this paper would be to compare the relative merits of the *outcomes* of different voting procedures whatever their strategic properties may be.

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# 1 APPENDIX

## 1.1 Proof of Proposition 3.1

We have $\overline{b}_n = \frac{\sum_{i=1}^{n}\omega_i b_i}{\sum_{i=1}^{n}\omega_i}$. The top expression has expectation $n\int_0^1 b\mu(b)dF(b)$ which, integrating by parts and using (4) in the text is equal to $n\mu\int_0^1 H(b)db$. Since $\sum_{i=a}^{n}\omega_i$ has expectation $n\mu$, the result follows from the strong law of large numbers. **Q.E.D.**

## 1.2 Proof of Proposition 3.2

For any $y \in [0,1]$, let $H_n(y+)$ denote the limit on the right of $H_n$ at $y$ which corresponds to the cumulative weight of individuals with bliss points strictly exceeding $y$. Since in equilibrium, those with bliss points strictly above $y_n^*$ vote 1 and only those with bliss points of at least $y_n^*$ may have a strictly positive vote,

$$H_n(y_n^*+) \leq y_n^* \leq H_n(y_n^*).$$

Thus if $\{\mathbf{H}_n\}$ converges uniformly to $H$ with probability 1, since $H$ is continuous, $\{\mathbf{y}_n\}$ must converge to $y^*$ with probability 1. In the remainder of the proof, we establish the uniform convergence of $\{\mathbf{H}_n\}$ to $H$ with probability 1.[12]

Let us first rewrite $\mathbf{H}_n(y)$ as

$$\mathbf{H}_n(y) = \frac{\frac{\sum_{i=1}^{n}\boldsymbol{\omega}_i I(b_i \geq y)}{n}}{\frac{\sum_{i=1}^{n}\boldsymbol{\omega}_i}{n}}.$$

where $I$ is an indicator function. Random variables $\boldsymbol{\omega}_i I(b_i \geq y)$ are drawn independently from an identical distribution with mean $\int_y^1 \mu(b)dF(b)$. Applying the strong law of large numbers to $\{\boldsymbol{\omega}_i I(b_i \geq y)\}$ and $\{\boldsymbol{\omega}_i\}$ yields that $\{\mathbf{H}_n(y)\}$ converges to $H(y)$ with probability 1 for all $y$ rational. Thus these countably many events are true with probability 1.

Finally, standard arguments may be used to establish that since $H_n$ is monotonically decreasing on $[0,1]$ for all $n$ and $H$ is continuous on $[0,1]$, pointwise convergence of $\{H_n\}$ to $H$ for a dense subset of $[0,1]$ containing 0 and 1, implies uniform convergence on $[0,1]$.[13]

[12] The following proof is adapted from that of Lemma 2 in Goldie (1977).

[13] See Lemma 1 in Goldie (1977) for an analogeous result.

## 1.3 Proof of Proposition 3.3

*(i)* We have

$$\nu = \int_0^1 H(x)dx = \int_0^{y^*} H(x)dx + \int_{y^*}^1 H(x)dx. \tag{9}$$

Now, if $H$ is convex, it is bounded above by the piecewise linear function taking values $1 + \frac{y^*-1}{y^*}x$ for $x \in [0, y^*]$ and $-\frac{y^*}{y^*-1} + \frac{y^*}{y^*-1}x$ for $x \in (y^*, 1]$. The result follows. A similar argument proves the result for $H$ concave.

To prove *(ii)*, let us show that

$$y^{\cdot *2} \leq \nu \leq 2y^* - y^{*2}$$

which clearly implies *(ii)*.

Once again, let us use equation (9) to express $\nu$ as the sum of two integrals. Using the definition of $y^*$, since $H$ is decreasing, $H$ is bounded below by the step function taking a value of $y^*$ on $[0, y^*]$ and 0 on $(y^*, 1]$. Similarly, it is bounded above by the step function taking a value of 1 on $[0, y^*]$ and $y^*$ on $(y^*, 1]$. The result follows. **Q.E.D.**

## 1.4 Proof of Proposition 4.1

The existence of a pure strategy equilibrium follows from theorem 2 in Rath (1992) which proves that a game with a continuum of players indexed by $t$ elements in $[0, 1]$ where all players share the same compact action space has a pure strategy Nash equilibrium. In our setting, we may view each player $i$ as a continuum of players indexed according to player $i's$ type. The union of these continua is isomorphic to the $[0, 1]$ interval. Because the utility function is continuous, the theorem applies.**Q.E.D.**

## 1.5 Proof of Proposition 4.2

*(i)* Continuity follows form strict concavity of $V$ with respect to $y$ and the theorem of the maximum.

$E_{\mathbf{S}^*_{-i}}(V'(\mathbf{S}^*_{-i} + w_i s, b_i)$ is strictly decreasing in $s$ on $[0, 1]$ due to assumption 4.1. Let $b$ and $b'$ be two bliss values with $b > b'$. If $s^*(b) = 1$ then we trivially have $s^*_i(b) \geq s^*_i(b')$. If

$s_i^*(b) < 1$ we must have $E_{\mathbf{S}^*_{-i}} V'(\mathbf{S}^*_{-i} + w_i\ s_i^*(b), b) \leq 0$. Since $V'$ is strictly increasing in $b$, it follows that $E_{\mathbf{S}^*_{-i}} V'(\mathbf{S}^*_{-i} + w_i\ s_i^*(b), b') < 0$. Hence $s_i^*(b) \geq s_i^*(b')$.

Since a sincere vote is a dominant strategy whenever $b_i = 0$ or 1, $s_i^*(0) = 0$ and $s_i^*(1) = 1$ and combined with statement *(i)* we deduce *(ii)* **Q.E.D.**

## 1.6 Proof of Lemma 4.1.

Let us first show that

**Step 1.** $\lim_{n\to\infty} \max_i [\overline{b}_{in} - \underline{b}_{in}] = 0$.

The proof proceeds by contradiction. Suppose the limit is not zero or does not exist. Then there exists $k > 0$ such that for any $N$ there exists $n > N$ with $\overline{b}_{in} - \underline{b}_{in} > k$ for some $i$. Let $g(S, b) = V'(S, b + k) - V'(S, b)$. Under assumption (4.1) $g$ is strictly positive on $[0, 1]^2$. Since $V'$ is uniformly continuous in $y$ and $\overline{w}_n$ tends to zero as $n$ tends to infinity, if $n$ is large enough, we have

$$E_{\mathbf{S}^*_{-in}} [V'(\mathbf{S}^*_{-in} + w_{in}, \underline{b}_{in} + k) - V'(\mathbf{S}^*_{-in}, \underline{b}_{in})] > 0. \tag{10}$$

Now, by definition of $\underline{b}_{in}$ we must have $E_{\mathbf{S}^*_{-in}} V'(\mathbf{S}^*_{-in}, \underline{b}_{in}) = 0$. On the other hand, since $\underline{b}_{in} + k \in (\underline{b}_{in}, \overline{b}_{in})$, the optimal vote of an agent of type $\underline{b}_{in} + k$ is strictly below 1 and we must have $E_{\mathbf{S}^*_{-in}} V'(\mathbf{S}^*_{-in} + w_{in}, \underline{b}_{in} + k) < 0$. This yields a contradiction.

To complete the proof we show

**Step 2.** $\lim_{n\to\infty} \max_{i,j} [\overline{b}_{in} - \overline{b}_{jn}] = 0$, and $\lim_{n\to\infty} \max_{i,j} [\underline{b}_{in} - \underline{b}_{jn}] = 0$.

Once again we proceed by contradiction. Suppose the first limit is not zero or does not exit. Then there exists $k > 0$ such that for any $N$ there exists $n > N$ with $\left|\overline{b}_{in} - \overline{b}_{jn}\right| > k$ for some $i, j$. *W.l.o.g* we may assume that $\overline{b}_{in} > \overline{b}_{jn}$. As in Step 1, $g(S, b)$ is strictly positive on $[0, 1]^2$ and, since $V'$ is uniformly continuous on $[0, 1]$ and $\overline{w}_n$ tends to zero as $n$ goes to infinity we have

$$E_{\mathbf{S}^*_{-jn}} V'(\mathbf{S}^*_{-jn} + w_{jn} + w_{in}, \overline{b}_{jn} + k) - V'(\mathbf{S}^*_{-jn} + w_{jn}, \overline{b}_{jn}) > 0, \tag{11}$$

for $n$ large enough. We also have

$$E_{\mathbf{S}^*_{-jn}} V'(\mathbf{S}^*_{-jn} + w_{jn} + w_{in}, \overline{b}_{jn} + k) \leq E_{\mathbf{S}^*_{-in}} V'(\mathbf{S}^*_{-in} + w_{in}, \overline{b}_{jn} + k) < 0, \tag{12}$$

where the first inequality follows from the concavity of $V$ with respect to $y$ and the second from the monotonicity of $V'$ with respect to $b$ and $\overline{b}_{jn} + k < \overline{b}_{in}$. This yields a contradiction. The second limit statement may be proved in a similar fashion. **Q.E.D.**

## 1.7 Proof of Proposition 4.3.

Let $\mathbf{C}_{1n} = \sum_{i=1}^{n} w_i I(b_i \geq \overline{b}_n)$ be the cumulative weight of agents with bliss points above $\overline{b}_n$. It is a random variable and we have

$$E[\mathbf{C}_{1n} - \widetilde{H}_n(\overline{b}_n) \quad | \quad \omega_i] = 0 \tag{13}$$

$$\text{and } var[\mathbf{C}_{1n} - \widetilde{H}_n(\overline{b}_n) \quad | \quad \omega_i] = \sum_{i=1}^{n} w_i^2 F(\overline{b}_n \mid \omega_i)(1 - F(\overline{b}_n \mid \omega_i)). \tag{14}$$

Similarly, let $\mathbf{C}_{2n} = \sum_{i=1}^{n} w_i I(b_i \leq \underline{b}_n)$ denote the cumulative weight of those who have bliss points less than $\underline{b}_n$. We have

$$E[\mathbf{C}_{2n} - [1 - \widetilde{H}_n(\underline{b}_n)] \quad | \quad \omega_i] = 0 \tag{15}$$

$$\text{and } var[\mathbf{C}_{2n} - [1 - \widetilde{H}_n(\underline{b}_n)] \quad | \quad \omega_i] = \sum_{i=1}^{n} w_i^2 F(\underline{b}_n \mid \omega_i)(1 - F(\underline{b}_n \mid \omega_i)). \tag{16}$$

Both variance expressions are bounded above by $\sum_{i=1}^{n} w_i^2$ which in turn is bounded above by $\overline{w}_n^2 + \overline{w}_n$. The latter upper bound is obtained by maximizing the sum expression with respect to the vector $(w_1, ..., w_n)$ subject to the constraint that it sums up to 1 and that each term is less than $\overline{w}_n > \frac{1}{n}$.[14] Hence both variances tend to 0 as $n$ goes to infinity and therefore $\mathbf{C}_{1n} - \widetilde{H}_n(\overline{b}_n)$ and $\mathbf{C}_{2n} - [1 - \widetilde{H}_n(\underline{b}_n)]$ tend to 0 in probability as $n$ goes to infinity. Using the mean value theorem we have

$$\widetilde{H}_n(\underline{b}_n) - \widetilde{H}_n(\overline{b}_n) \leq \beta(\overline{b}_n - \underline{b}_n)$$

where $\beta$ is the uniform bound on the conditional density $f(b \mid \omega)$ (such a uniform bound exists since the unconditional mean of bliss points is assumed to be finite). Then Lemma (4.1) implies that $\lim_{n\to\infty}(\widetilde{H}_n(\overline{b}_n) - \widetilde{H}_n(\underline{b}_n)) = 0$. Thus $\mathbf{C}_{1n} - \widetilde{H}_n(\underline{b}_n)$ must also tend to 0 in

[14]The expression $\overline{w}_n^2 + \overline{w}_n$ is actually an upper bound on the maximal value of the sum. Because the sum is strictly convex, it is maximized by a corner solution in which at most one weight is strictly between zero and $\overline{w}_n$. The upper bound is obtained by noting that the largest number of individuals who may be awarded the largest weight is bounded above by $\frac{1}{\overline{w}_n}$ and the remaining weight is bounded above by $\overline{w}_n$.

probability. Now if $\mathbf{y}_n^{**}$ is the equilibrium outcome, we have

$$\mathbf{C}_{1n} \leq \mathbf{y}_n^{**} \leq 1 - \mathbf{C}_{2n}. \tag{17}$$

Rearranging and taking the limit in probability yields

$$\mathrm{p}\lim_{n\to\infty} [\mathbf{y}_n^{**} - \widetilde{H}_n(\underline{b}_n)] = 0. \tag{18}$$

Hence, to prove the result, it is enough to show that

$$\lim(\underline{b}_n - \widetilde{y}_n) = 0.$$

We first show

**Claim 1.1** *There exists a sequence of strictly positive numbers* $\{\eta_n\}$ *such that* $\widetilde{H}_n(\underline{b}_n) \leq \underline{b}_n + \eta_n$ *for all* $n$, *and* $\lim_{n\to\infty} \eta_n = 0$.

Suppose to the contrary that there exists a $K > 0$ and a subsequence $\underline{b}_{n_k}$ such that $\widetilde{H}_n(\underline{b}_{n_k}) > \underline{b}_{n_k} + K$ for all $n_k$. Then

$$V(\widetilde{H}_{n_k}(\underline{b}_{n_k}) + s, \underline{b}_{n_k} + \Delta) < V(\widetilde{H}_{n_k}(\underline{b}_{n_k}), \underline{b}_{n_k} + \Delta) \tag{19}$$

for any $s > 0$ and any $\Delta \in (0, K)$. Furthermore using (18), by a standard bounded convergence argument we have

$$\lim_{n_k\to\infty} E_{\mathbf{y}_{n_k}^{**}} \left[V(\mathbf{y}_{n_k}^{**} + s, \underline{b}_{n_k} + \Delta)\right] - V(\widetilde{H}_{n_k}(\underline{b}_{n_k}) + s, \underline{b}_{n_k} + \Delta) = 0 \tag{20}$$

for all $s \in [0, 1]$. Thus, using equation (19) and (20), for $n_k$ sufficiently large,

$$E_{\mathbf{y}_{n_k}^{**}} \left[V(\mathbf{y}_{n_k}^{**} + s, \underline{b}_{n_k} + \Delta)\right] < E_{\mathbf{y}_{n_k}^{**}} \left[V(\mathbf{y}_{n_k}^{**}, \underline{b}_{n_k} + \Delta)\right] \tag{21}$$

It is optimal for any agent of type $\underline{b}_{n_k} + \Delta$ to vote 0. Thus $\underline{b}_{n_k} \geq \underline{b}_{n_k} + \Delta$, which contradicts $\Delta > 0$.

**Claim 1.2** *There exists a sequence of strictly positive numbers* $\{\underline{\epsilon}_n\}$ *such that* $\widetilde{y}_n \leq \underline{b}_n + \underline{\epsilon}_n$ *for all* $n$ *and* $\lim_{n\to\infty} \underline{\epsilon}_n = 0$.

Using the definition of $\widetilde{y}_n$ and Claim A.1 we have

$$\widetilde{H}_n(\underline{b}_n) - \widetilde{H}_n(\widetilde{y}_n) + \widetilde{y}_n - \underline{b}_n < \eta_n. \tag{22}$$

If $\underline{b}_n \geq \widetilde{y}_n$, claim A.2 trivially holds. If $\underline{b}_n < \widetilde{y}_n$, the left hand side of (22) is strictly positive since $\widetilde{H}_n$ is decreasing. If this is the case for some subsequences $\{\underline{b}_{n_k}\}$ and $\{\widetilde{y}_{n_k}\}$, then the left-hand side of (22) must tend to 0 when $n_k$ goes to infinity since $\lim_{n_k \to \infty} \eta_{n_k} = 0$. Since both $\widetilde{y}_n - \underline{b}_n$ and $\widetilde{H}_n(\underline{b}_n) - \widetilde{H}_n(\widetilde{y}_n)$ are positive, they must both tend to zero. Thus claim A.2 holds.

Symmetrically it can be shown that there exists a sequence of strictly positive numbers$\{\overline{\varepsilon}_n\}$ such that $\widetilde{y}_n \geq \overline{b}_n + \overline{\varepsilon}_n$ for all $n$ and $\lim_{n\to\infty} \overline{\varepsilon}_n = 0$. Using Lemma (4.1) we have $\lim_{n\to\infty}(\underline{b}_n - \widetilde{y}_n) = 0$. **Q.E.D.**

## 1.8 Proof of Lemma 4.2

We may rewrite $\overline{\mathbf{w}}_n$ as $\left[ n \frac{1}{n} \sum_{i=1}^{n} \frac{\boldsymbol{\omega}_i}{\max_i \boldsymbol{\omega}_i} \right]^{-1}$. The random variable $\frac{\boldsymbol{\omega}_i}{\max_i \boldsymbol{\omega}_i}$ takes on values in [0,1] and has a finite and strictly positive expectation. The results follows from applying the strong law of large numbers.

## 1.9 Proof of Proposition 4.4

By an argument similar to that of the proof of Proposition 3.2, it can be shown that $\widetilde{\mathbf{H}}_n$ converges to $H$ uniformly with probability 1. Indeed $\widetilde{\mathbf{H}}_n(y) = \frac{[\sum_i \boldsymbol{\omega}_i [1-F(y|\boldsymbol{\omega}_i]/n]}{[\sum_i \boldsymbol{\omega}_i/n]}$. Variables at the top are i.i.d with mean $H_n(y)\overline{\mu}$, while variables at the bottom are i.i.d with mean $\overline{\mu}$. Then the strong law of large numbers may be used in a similar fashion as in the proof of Proposition 3.2 to establish uniform convergence. Then the result follows. **Q.E.D.**